On Polars of Plane Branches

A. Hefez (1), M. E. Hernandes (2), M. F. Hernández Iglesias (2)

((1) Universidade Federal Fluminense, (2) Universidade Estadual de Maringá)

It is well known that the equisingularity class of the general polar of a plane branch is not the same for all branches in a given equisingularity class, but it is the same for sufficiently general ones (cf. [C2]) and depends upon the analytic type of the branch. The aim of this paper is to go beyond generality and show how one could describe the equisingularity classes of (general) polars of all branches in a given equisingularity class, making use of the analytic classification of branches as described in [HH3]. We will show how this works in some particular equisingularity classes for which one has the complete explicit analytic classification, and in particular for all branches of multiplicity less or equal than four, based on the classification given in [HH2].

1 Introduction

Let $f \in \mathbb{C}\{x, y\}$ be a convergent power series in two variables over the field of complex numbers with no multiple factors and such that $f(0,0) = 0$. Notice that because of finite determinacy the same results hold without changes if the power series are only formal. We consider the germ of curve $(f)\colon f = 0$ at the origin of $\mathbb{C}^2$, which determines f up to multiplication by a unit. An irreducible germ of curve will be called a *branch*. We will say that the germs of curves (f) and (g) are *analytically equivalent* if there exists a germ of analytic diffeomorphism at the origin of $\mathbb{C}^2$ that transforms one germ into the other. In terms of equations, this translates into the fact that there exist a unit u and an automorphism ϕ of $\mathbb{C}\{x, y\}$ such that $g = u\phi(f)$. In this case, we also say that the functions f and g are *contact equivalent*. The *equisingularity class* of a curve is its equivalence class under transformations by germs of homeomorphisms at the origin of $\mathbb{C}^2$.

The polar curve of f in the direction $(a\colon b) \in \mathbb{P}^1_{\mathbb{C}}$ is the germ of curve defined by the equation $af_x + bf_y = 0$. It is known (cf. [C3] Theorem 7.2.10) that, except for a finite set of directions, the polar is reduced and its equisingularity class is constant, although its analytic type depends essentially upon the direction $(a : b)$, as we will see in an example at the end of the

paper. Also, the equisingularity class of the general polar of f is constant in the contact class of f (cf. [C3] Corollary 8.5.8), but it is not constant in the equisingularity class of f, as one can easily check by considering for example the curves $y^3 - x^{11}$ and $y^3 - x^{11} + x^8 y$ (cf. [P]). So, the topological type of the polar of a given curve is not determined only by the topological type of the curve, but it is determined by its analytical type. In the next section we will see to what extent the analytic type of the curve will influence the topology of its polar.

We refer to [Z] for the definitions and basic results we will use in the sequel. It is a classical result that the equisingularity class of a reduced curve given by $f = f_1 \cdots f_r$, where th f_i are irreducible, is determined by the semigroups of the f_i's and their mutual intersection numbers $\mathrm{I}(f_i, f_j)$, for $i \neq j$. A semigroup of values Γ of a branch will be given by its minimal set of generators $\Gamma = \langle v_0, v_1, \dots, v_g \rangle$ and the integer g will be called the *genus* of the branch. Such a semigroup has a conductor c and the equisingularity class it determines may be parametrized by a constructible set $\mathcal{E}$ in $\mathbb{C}^{c-v_1-1}$, whose points are the coefficients of the Newton-Puiseux parametrization

$$x(t) = t^{v_0}, \quad y(t) = t^{v_1} + \sum_{i=v_1+1}^{c-1} c_i t^i,$$

in the sense that any element in the equisingularity class is analytically equivalent to one with a Newton-Puiseux parametrization as above.

Given an equisingularity class of irreducible curves, it was proved in [HH1] and [HH3] that the parameter space $\mathcal{E}$ may be decomposed into a finite union of disjoint constructible sets $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_r$, where on each $\mathcal{E}_\ell$, the set Λ_ℓ of values of Kähler differentials on the corresponding curve, which is an analytic invariant of the curve, is fixed.

Since Γ has a conductor and $\Gamma \setminus \{0\} \subset \Lambda_\ell$, the set Λ_ℓ is determined by the finite set $\Lambda_\ell \setminus \Gamma$. If this last set is not empty, the natural number λ associated to a curve represented by a point of $\mathcal{E}_\ell$, defined as

$$\lambda = \min(\Lambda_\ell \setminus \Gamma) - v_0,$$

is an analytic invariant known as the *Zariski invariant* of the curve.

We will now recall a result that shows that the elements of $\mathcal{E}_\ell$ admit a normal form.

Normal Forms Theorem (cf. [HH3]) *If $\mathcal{C}$ is a curve corresponding to a point in $\mathcal{E}_\ell$, then either $\mathcal{C}$ is analytically equivalent to a curve with parametrization (t^{v_0}, t^{v_1}), when $\Lambda_\ell \setminus \Gamma = \emptyset$, or to a curve with a parametrization of the form*

$$x = t^{v_0}, \quad y = t^{v_1} + t^\lambda + \sum_i c_i t^i,$$

where the summation is over all indices i greater than λ and do not belong to the set $\Lambda_\ell - v_0$. Moreover, two curves $\mathcal{C}$, with a parametrization as above,

and $\mathcal{C}'$ with a similar parametrization but with coefficients (c_i') instead of (c_i), are analytically equivalent if and only if there exists a complex number ζ such that $\zeta^{\lambda - v_1} = 1$ and for all i, one has $c_i = \zeta^{i-v_1} c_i'$.

At this point it is natural to ask if the equisingularity class of the general polar is constant on each stratum $\mathcal{E}_\ell$.

We will see in the next section that the answer may be negative, but positive for a general member of each irreducible component of the family. This was shown in [C2] in the particular case of the whole space $\mathcal{E}$.

For the convenience of the reader, we will state a well known result about Newton non-degenerate plane curve singularities that will be needed in our analysis.

We say that a reduced curve (f), where $f = \sum_{ij} a_{ij} x^i y^j$ is *Newton non-degenerate* if x and y do not divide f and for any side L of its Newton polygon, the polynomial $f_L = \sum_{(i,j)\in L} a_{ij} x^i y^j$ has no critical points outside the curve $xy = 0$ (cf. [K]). Since f_L is a quasi-homogeneous polynomial, we may rephrase the Newton non-degeneration as follows:

Let $P_k = (i_k, j_k)$ and $P_{k+1} = (i_{k+1}, j_{k+1})$ be the extremal points of L and define

$$p_L(z) = z^{-j_{k+1}} f_L(1, z).$$

Then one has that f_L has no critical points outside the curve $xy = 0$ if and only if p_L has no multiple roots.

The following result (cf. [O] or [BLP]) will describe the equisingularity class of a Newton non-degenerate curve (g) such that (x) does not belong to its tangent cone:

There is a decomposition (g_i), $i = 1, \dots, s$, of (g), such that the Newton polygon of (g_i) is the segment $[(0, n_i); (m_i, 0)]$, with $n_i = I(g_i, x)$ and $m_i = I(g_i, y)$ and $1 \le d_1 < d_2 < \cdots < d_s \le \infty$, where $d_i = \frac{m_i}{n_i}$, and $d_s = \infty$ if $(g_s) = (y)$. When (g) is Newton non-degenerate, then (g_i) has $r_i = gcd(n_i, m_i)$ branches given by Newton-Puiseux parametrizations

$$(g_i)_j \colon y_{i,j}^{\frac{n_i}{r_i}} = a_{i,j} x^{\frac{m_i}{r_i}} + \cdots, \quad 1 \le j \le r_i, \quad \text{with} \quad a_{i,j} \ne a_{i,j'}, \quad \text{for} \quad j \ne j',$$

and

$$\mathrm{I}((g_i)_j, (g_{i'})_{j'}) = \inf\{d_i, d_{i'}\} \mathrm{I}((g_{i'})_{j'}, x) \mathrm{I}((g_i)_j, x).$$

This paper contains some results from the PhD thesis of the third author under the supervision of the other two (cf. [HI]).

2 Polars and Normal Forms

We will see in the following example how the Normal Forms Theorem may be used to describe the equisingularity classes of the general polars of all members of a given equisingularity class.

Example 1. Let $\Gamma = \langle 5, 12\rangle$. The Normal Forms Theorem, together with the algorithm to compute normal forms in [HH4], gives us the complete classification of the curves in the equisingularity class determined by Γ that we summarize in the following table:

Normal Form	$\Lambda_\ell \setminus \Gamma$
1. (t^5, t^{12})	$\emptyset$
2. $(t^5, t^{12} + t^{38})$	$\{43\}$
3. $(t^5, t^{12} + t^{33})$	$\{38, 43\}$
4. $(t^5, t^{12} + t^{28})$	$\{33, 38, 43\}$
5. $(t^5, t^{12} + t^{26} + ct^{28}),\ c \neq 0$	$\{31, 38, 43\}$
6. $(t^5, t^{12} + t^{26} + ct^{33})$	$\{31, 43\}$
7. $(t^5, t^{12} + t^{23} + ct^{26})$	$\{28, 33, 38, 43\}$
8. $(t^5, t^{12} + t^{21} + ct^{23} + dt^{28})$	$\{26, 31, 38, 43\}$
9. $(t^5, t^{12} + t^{18} + ct^{21} + dt^{26})$	$\{23, 28, 33, 38, 43\}$
10. $(t^5, t^{12} + t^{16} + ct^{18} + dt^{23})$	$\{21, 26, 31, 33, 38, 43\}$
11. $(t^5, t^{12} + t^{14} + ct^{16} + dt^{18} + et^{23}),\ c \neq \frac{13}{12},\ d \neq \frac{4c^2-1}{3}$	$\{19, 26, 31, 33, 38, 43\}$
12. $(t^5, t^{12} + t^{14} + ct^{16} + (\frac{4c^2-1}{3})t^{18} + dt^{23} + et^{28}),\ c \neq \frac{13}{12}$	$\{19, 26, 31, 38, 43\}$
13. $(t^5, t^{12} + t^{14} + \frac{13}{12}t^{16} + ct^{18} + dt^{21}),\ c \neq \frac{133}{108}$	$\{19, 28, 31, 33, 38, 43\}$
14. $(t^5, t^{12} + t^{14} + \frac{13}{12}t^{16} + \frac{133}{108}t^{18} + ct^{21} + dt^{23}),\ d \neq \frac{34c}{11}$	$\{19, 31, 33, 38, 43\}$
15. $(t^5, t^{12} + t^{14} + \frac{13}{12}t^{16} + \frac{133}{108}t^{18} + ct^{21} + \frac{34}{11}ct^{23} + dt^{28}),$ $d \neq \frac{81c^2}{32} + \frac{5225}{559872}$	$\{19, 31, 38, 43\}$
16. $(t^5, t^{12} + t^{14} + \frac{13}{12}t^{16} + \frac{133}{108}t^{18} + ct^{21} + \frac{34}{11}ct^{23} +$ $(\frac{81c^2}{32} + \frac{5225}{559872})t^{28} + dt^{33})$	$\{19, 31, 43\}$
17. $(t^5, t^{12} + t^{13} - \frac{1}{2}t^{14} + ct^{16} + dt^{21} + et^{26})$	$\{18, 23, 28, 33, 38, 43\}$
18. $(t^5, t^{12} + t^{13} + ct^{14} + dt^{16} + et^{21}),\ c \neq -\frac{1}{2}$	$\{18, 23, 28, , 31, 33, 38, 43\}$

Table 2.1: The normal forms of the equisingularity class of $\langle 5, 12\rangle$

Now, with the help of the Maple software we obtain the implicit equations of the curves given by the parametrization in each row of the table, then we exhibit their polars and analyze the equisingularity classes of these polars. In what follows, the symbols u_1, u_2, u_3 and u_4 represent units in $\mathbb{C}\{x, y\}$, with $u_i(0,0) = 1$, for $1 \leq i \leq 4$, not necessarily the same in all cases.

1. $af_x + bf_y = 5by^4 - 12ax^{11}$.
2. $af_x + bf_y = 5by^4 - 50ax^9y^3 - 15bx^{10}y^2 + 100ax^{19}y - 12ax^{11} + 5bx^{20} - 38ax^{37}$.
3. $af_x + bf_y = 5by^4 - 45ax^8y^3 - 15bx^9y^2 + 90ax^{17}y - 12ax^{11} + 5bx^{18} - 33ax^{32}$.
4. $af_x + bf_y = 5by^4 - 40ax^7y^3 - 15bx^8y^2 + 80ax^{15}y - 12ax^{11} + 5bx^{16} - 28ax^{27}$.
5. $af_x + bf_y = 5by^4 - 40acx^7y^3 - 15bcu_1x^8y^2 - 10bu_2x^{10}y - 12au_3x^{11}$.
6. $af_x + bf_y = 5by^4 - 45acx^8y^3 - (15bc + 50a)u_1x^9y^2 - 10bu_2x^{10}y - 12au_3x^{11}$.
7. $af_x + bf_y = 5by^4 - 35ax^6y^3 - 15bu_1x^7y^2 - 10bcu_2x^{10}y - 12au_3x^{11}$.
8. $af_x + bf_y = 5by^4 - 35acu_1x^6y^3 - 15bcu_2x^7y^2 - 10bu_3x^9y - 12au_4x^{11}$.
9. $af_x + bf_y = 5by^4 - 30ax^5y^3 - 15bu_1x^6y^2 - 10bcu_2x^9y - 12au_3x^{11}$.

The polar of any one of the curves in the families $(1) - (9)$ has Newton polygon with only one side $L = [(0,4); (11,0)]$ that supports only its extremal points associated to the monomials $5by^4$ and $-12ax^{11}$. This implies

that all (general) polars are Newton non-degenerate, so their Newton polygons determine their equisisingularity classes, which in this case is given by only one branch with semigroup $\langle 4, 11\rangle$.

10. $af_x + bf_y = 5by^4 - 30acu_1x^5y^3 - 15bcu_2x^6y^2 - 10bu_3x^8y - 12au_4x^{11}$.

In this case, the Newton polygon of the polar has two sides:

$L_1 = [(0,4);(8,1)]$, that supports only its extremal points associated to the monomials $5by^4$ and $-10bx^8y$.

$L_2 = [(8,1);(11,0)]$, that supports only its extremal points associated to the monomials $-10bx^8y$ and $-12ax^{11}$.

Again, the polar of any curve belonging to this family is Newton non-degenerate with two branches: g_1 with semigroup $\langle 3, 8\rangle$ and g_2 smooth such that $\mathrm{I}(g_1, g_2) = 8$.

11.

$$\begin{aligned} af_x + bf_y &= 5by^4 - 30a(c+d)u_1x^5y^3 - 15b(c+d)u_2x^6y^2 \\ &\quad -10b(1+c)u_3x^8y - 5bu_4x^{10}. \end{aligned}$$

12.

$$\begin{aligned} af_x + bf_y &= 5by^4 + 10a(1-3c-4c^2)u_1x^5y^3 + 5b(1-3c-4c^2)u_2x^6y^2 \\ &\quad -10b(1+c)u_3x^8y - 5bu_4x^{10}. \end{aligned}$$

13.

$$\begin{aligned} af_x + bf_y &= 5by^4 - \tfrac{5}{2}a(13+12c)u_1x^5y^3 - \tfrac{5}{4}b(13+12c)u_2x^6y^2 \\ &\quad -\tfrac{125}{6}bu_3x^8y - 5bu_4x^{10}. \end{aligned}$$

14. $af_x + bf_y = 5by^4 - \frac{625}{9}au_1x^5y^3 - \frac{625}{18}bu_2x^6y^2 - \frac{125}{6}bu_3x^8y - 5bu_4x^{10}$.

15. $af_x + bf_y = 5by^4 - \frac{625}{9}au_1x^5y^3 - \frac{625}{18}bu_2x^6y^2 - \frac{125}{6}bu_3x^8y - 5bu_4x^{10}$.

16. $af_x + bf_y = 5by^4 - \frac{625}{9}au_1x^5y^3 - \frac{625}{18}bu_2x^6y^2 - \frac{125}{6}bu_3x^8y - 5bu_4x^{10}$.

17. $af_x + bf_y = 5by^4 - 25au_1x^4y^3 - 15bu_2x^5y^2 + 5b\left(\frac{1-4c}{2}\right)u_3x^8y + \frac{15}{2}bu_4x^{10}$.

The Newton polygon of the polars of any member of the families (11) – (17) has only one side $L = [(0,4);(10,0)]$, that supports just its extremal points associated to the monomials y^4 and x^{10} with some non-zero coefficients that do not depend upon the parameters c, d and e. Therefore, these polars are Newton non-degenerate, so they have two branches with semigroup $\langle 2, 5\rangle$ that intersect with multiplicity 10.

18.

$$\begin{aligned} af_x + bf_y = \ & 5by^4 - 25au_1x^4y^3 - 15bu_2x^5y^2 \\ & -10[b(c^2+c+d)x^8 + (b(d^2+e) + 5a(c-1))u_3x^9]y \\ & +5b(1-c)x^{10} + (-5b(c^3+dc+d) - 12au_4x^{11}. \end{aligned}$$

This is the only stratum in which the equisingularity class of the polars will depend upon the parameters in $\mathcal{E}_\ell$ to which it belongs.

(i) If $c \neq 1$, then the Newton polygon of the polar has the only side $L = [(0,4);(10,0)]$, that supports the points associated to the monomials $5by^4$, $-15bx^5y^2$ and $-5b(c-1)x^{10}$. In this case, $p_L(z) = z^4 - 3z^2 - (c-1)$ whose

discriminant is $-16(c-1)(5+4c)^2$. So, the polar is Newton non-degenerate if and only if $c \neq -\frac{5}{4}$. In this case, the polar has two branches with semigroup $\langle 2,5\rangle$ that intersect with multiplicity 10.

If $c = -\frac{5}{4}$, the parametrization of the polar is given by

$$x = \frac{3^{20}}{2^{21}}(16d+5)^2 t^4, \quad y = \frac{3^{38}}{2^{53}}(16d+5)^5 t^{10} + \frac{3^{30}}{2^{60}}(16d+5)^6 t^{11} + \cdots .$$

Therefore, when $d \neq -\frac{5}{16}$, the members of the family $\mathcal{E}_{18}$ for which $c = -\frac{5}{4}$ have irreducible polars of genus 2 with semigroup $\langle 4,10,21\rangle$, and when $d = -\frac{5}{16}$, they have polars with two branches with parametrizations

$$x_i = \frac{3}{2}t^2, \quad y_i = \frac{27}{8}t^5 + (-1)^i \frac{27}{640b}(256ab - 125b)^{\frac{1}{2}} t^6 + \cdots , \quad i = 1,2,$$

that is, branches with semigroup $\langle 2,5\rangle$ and with intersection multiplicity 11.

(ii) If $c = 1$, then the polar is given by

$$af_x + bf_y =$$

$$5by^4 - 25ax^4u_1y^3 - 15bx^5u_2y^2 - 10b(d+2)x^8u_3y - (12a+5b+10bd)x^{11}u_4.$$

An easy computation shows that its Newton polygon has two sides:

$L_1 = [(0,4);(5,2)]$, that supports only its extremal points associated to the monomials $5by^4$ and $-15bx^5y^2$; and

$L_2 = [(5,2);(11,0)]$, that supports only its extremal points associated to the monomials $-15bx^5y^2$ and $-(12a+5b+10bd)x^{11}$.

Therefore, any curve in this family with $c = 1$ has, for general values of a and b, a Newton non-degenerate polar with a branch p with semigroup $\langle 2,5\rangle$ and two non-singular branches g_1 and g_2 such that $\mathrm{I}(p,g_i) = 5$ and $\mathrm{I}(g_1,g_2) = 3$.

Remark The stratum $\mathcal{E}_{18}$ gives us an example in which the equisingularity class of the general polar of its members is not constant. It also gives us a somewhat unexpected example of a family of curves of genus 1 such that its general member has a general polar of genus 2.

What is remarkable is that the analytic classification of the branches in this equisingularity class allowed us to describe the equisingularity classes of all general polars of its members.

Although, as we saw in the above example, the topological type of the polar may be not constant in a given stratum $\mathcal{E}_\ell$, it is constant in an open dense set of each irreducible component of the stratum, as we will show in general in the sequel.

In fact, for the stratum associated to $\Lambda = \Gamma \setminus \{0\}$, the result follows easily. Let us consider a normal form in an equisingularity class parametrized by $\mathcal{E}_\ell$ associated to a set of values of differentials $\Lambda_\ell \neq \Gamma \setminus \{0\}$. Putting $v_0 = n$ and $v_1 = m$, from the Normal Forms Theorem, we have

$$x = t^n, \quad y = t^m + t^\lambda + \sum_{\substack{i>\lambda \\ i \notin \Lambda_\ell - n}} c_i t^i.$$

The implicit equations of these curves are given by Weierstrass polynomials

$$f = y^n + a_2(x)y^{n-2} + a_3(x)y^{n-3} + \cdots + a_{n-1}(x)y + a_n(x),$$

where the coefficients $a_j(x)$ are polynomials in the variables c_i and such that $\mathrm{ord}_x(a_j(x)) > j$ and $\mathrm{ord}_x a_n(x) = m$.

Therefore, the polars of the curves in $\mathcal{E}_\ell$ are given by the family

$$\begin{aligned} P(f) &= af_x + bf_y \\ &= bny^{n-1} + aa_2'(x)y^{n-2} + \big(ba_2(x)(n-2) + aa_3'(x)\big)y^{n-3} + \cdots \\ &\quad + \big(b(n-2)a_{n-2}(x) + aa_{n-1}'(x)\big)y + ba_{n-1}(x) + aa_n'x. \end{aligned}$$

We will now show that in a dense open Zariski set in any irreducible component of $\mathcal{E}_\ell$ the value of the Milnor number of $P(f)$ is constant.

From the equation of $P(f)$ we have that

$$\begin{aligned} P(f)_x &= aa_2''(x)y^{n-2} + \big(aa_3''(x) + b(n-2)a_2'(x)\big)y^{n-3} + \cdots + \\ &\quad \big(aa_{n-1}''(x) + 2ba_{n-2}'(x)\big)y + aa_n''(x) + ba_{n-1}'(x), \quad \text{and} \end{aligned}$$

$$\begin{aligned} P(f)_y &= bn(n-1)y^{n-2} + a(n-2)a_2'(x)y^{n-3} + \cdots + \\ &\quad \big(aa_{n-1}'(x) + b(n-2)a_{n-2}(x)\big). \end{aligned}$$

Therefore, one has that $P(f)_y$ is a constant times a Weierstrass polynomial in y and $P(f)_x \in \mathbb{C}\{x\}[y]$, hence their intersection multiplicity, which is the Milnor number of $P(f)$, is the order in x of their resultant R_y in y. Because $R_y \not\equiv 0$ since the generic polar of f is reduced, we have for every irreducible component $\mathcal{E}_{\ell,j}$ of $\mathcal{E}_\ell$ that

$$R_y\big(P(f)_x, P(f)_y\big) = A_j x^{\nu_j} + \textit{higher order terms},$$

where A_j is a non-zero polynomial in a, b and the c_i (the coefficients in the normal forms) and homogeneous in a and b. So, there exists a Zariski open set in $\mathcal{E}_{\ell,j}$, where at each point this polynomial A_j in a and b is not identically zero, hence the Milnor number of the general polar of the corresponding curve is constant ($= \nu_j$) in this open set. From the Lê-Ramanujan Theorem [LR], we obtain the following result:

Theorem 1. *The equisingularity class of the polar of curves in $\mathcal{E}_\ell$ is constant in an open dense Zariski subset of any of its irreducible components.*

3 Polars of branches up to multiplicity four

We will now give a detailed description of the equisingularity classes of the polars of branches of multiplicity less or equal than four. This will be carried out by using the classification done by the first two authors in [HH3]. Observe that the polar of a branch of multiplicity 2 is a smooth branch, so we have only to treat the cases of multiplicities three and four.

3.1 Multiplicity three

For multiplicity three curves, there is only one analytic representative in each stratum which is determined by Zariski's λ invariant, as shown in the table below:

$\Gamma = \langle 3, \beta \rangle; \qquad \beta = 3q + \varepsilon, \quad \varepsilon = 1,\ 2$
$x = t^3,\ \ y = t^\beta$
$x = t^3,\ \ y = t^\beta + t^{\beta+\varepsilon+3k},\ 0 \leq k \leq q-2$

For the case of the monomial curve $x = t^3,\ y = t^\beta$, we have that the polar curve has $d = \gcd(2, \beta-1)$ branches. When $d = 1$, the branch has semigroup $\langle 2, \beta-1 \rangle$ and when $d = 2$, the two branches are smooth and their intersection multiplicity is $\frac{\beta-1}{2}$.

In the case of the second row of the above table, the implicit equation of the curve is $f = y^3 - 3x^{2q+k+\epsilon}y - x^\beta - x^{\beta+\epsilon+3k}$ and the generic polar curve is

$$af_x + bf_y = 2by^2 - 3a(2q+k+\epsilon)x^{2q+k+\epsilon}y - 3bux^{2q+k+\epsilon},$$

where u is a unit. After a direct computation, we see that the equisingularity class of the polar may be described by the following table:

$2q + k + \epsilon = 2I + 1$	One branch with semigroup $\langle 2, 2q + k + \epsilon \rangle$.
$2q + k + \epsilon = 2I$	Two smooth branches with intersection multiplicity I.

3.2 Multiplicity four, genus one

A curve of multiplicity 4 may have genus one or two. For the genus one case, we have the following normal forms:

Normal form	$\Lambda \setminus \langle 4, m\rangle$
1. $y(t) = t^m$	$\emptyset$
2. $y(t) = t^m + t^{3m-4j} + a_1 t^{2m-4(j-[\frac{m}{4}]-1)} + \cdots$ $+a_{j-[\frac{m}{4}]-2} t^{2m-8}, \quad 2 \leq j \leq [\frac{m}{2}]$	$\{3m-4s; 1 \leq s \leq j-1\}$
3. $y(t) = t^m + t^{2m-4j} + a_k t^{3m-(4[\frac{m}{4}]+j+1-k)} + \cdots$ $+a_{j-[\frac{m}{4}]-2} t^{3m-4([\frac{m}{4}]+3-k)}$ $a_k \neq 0, \quad 2 \leq j \leq [\frac{m}{4}], \;\; 1 \leq k \leq [\frac{m}{4}]-j$	$\{2m-4s; 1 \leq s \leq j-1\} \cup$ $\{3m-4s; 1 \leq s \leq [\frac{m}{4}]+1-k\}$
4. $y(t) = t^m + t^{2m-4j} + a_{[\frac{m}{4}]-j+1} t^{3m-8j}$ $+a_{[\frac{m}{4}]-j+2} t^{3m-4(2j-1)} + \cdots + a_{[\frac{m}{4}]-1} t^{3m-4(j+2)},$ $a_{[\frac{m}{4}]-j+1} \neq \frac{3m-4j}{2m}, \quad 2 \leq j \leq [\frac{m}{4}]$	$\{2m-4s; 1 \leq s \leq j-1\} \cup$ $\{3m-4s; 1 \leq s \leq j\}$
5. $y(t) = t^m + t^{2m-4j} + \frac{3m-4j}{2m} t^{3m-8j} +$ $a_{[\frac{m}{4}]-j+2} t^{3m-4(2j-1)} + \cdots + a_{[\frac{m}{4}]} t^{3m-(j+1)},$ $2 \leq j \leq [\frac{m}{4}]$	$\{2m-4s; 1 \leq s \leq j-1\} \cup$ $\{3m-4s; 1 \leq s \leq j-1\}$

Table 3.1: Normal forms for multiplicity four and genus one

First Normal Form (monomial curves)

In this case, the equation of the curve is $y^4 - x^m = 0$, so its polar is $4by^3 - amx^{m-1}$, that has $d = \gcd(3, m-1)$ branches. If $d = 1$, the branch has semigroup $\langle 3, m-1\rangle$ and when $d = 3$, the three branches are smooth with mutual intersection multiplicity equal to $\frac{(m-1)}{3}$.

Second Normal Form

This is the more complicated case. The implicit equation of the curve is

$$f = y^4 - S_1(x)y^3 + S_2(x)y^2 - S_3(x)y + S_4(x) = 0,$$

where $S_r(x)$ is the r-th symmetric polynomial computed in $y(\varepsilon^l t)$, $l = 0, 1, 2, 3$, with ε a primitive fourth root of 1 and where we have replaced t^4 by x.

From the definition of $y(t)$, it is clear that $S_1 = 0$. To determine the Newton polygon of the polar, it is sufficient to consider in the polynomial $S_r(x)$, $2 \leq r \leq 3$, the monomial which determines its multiplicity.

(I) We first consider the case $a_1 = a_2 = \ldots = a_{j-[\frac{m}{4}]-2} = 0$.

For each fixed j, we have $f = y^4 - 4x^{m-j}y^2 - x^m + 2x^{2m-2j} - x^{3m-4j}$. Therefore,

$$af_x + bf_y = 4by^3 - 4a(m-j)x^{m-j-1}y^2 - 8bx^{m-j}y - amx^{m-1}u,$$

where $u \in \mathbb{C}\{x\}$ with $u(0) = 1$.

We have the following cases:

i. Case $\frac{2}{m-j} < \frac{1}{j-1}$.

In this case, the Newton polygon of the polar has only one side L containing only its end points $(0,3)$ and $(m-1,0)$, associated to monomials of the polar. The polynomial associated to the Newton polygon is $p_L(z)=4bz^3-am$. Then for a and b generic, $p_L(z)$ has three distinct roots $\{z_1,z_2,z_3\}$. Therefore the polar has:

a) One branch with semigroup $\langle 3,m-1\rangle$, if $\gcd(3,m-1)=1$.

b) Three smooth branches with parametrizations: $(t,z_it^{\frac{m-1}{3}}+\cdots)$, and mutual intersection numbers $\frac{m-1}{3}$, if $\gcd(3,m-1)=3$.

ii. Case $\frac{2}{m-j}>\frac{1}{j-1}$.

In this case, the Newton polygon of the polar has two sides L_1 and L_2, each one with only its end points associated to monomials of the polar. The associated polynomials are $p_{L_1}(z)=4bz^2-8b$, and $p_{L_2}(z)=-8bz-am$. Then, we have that

a) Associated to L_1 there is one branch p_1 with semigroup $\langle 2,m-j\rangle$ and parametrization $x=t^2$ $y=\sqrt{2}t^{m-j}+\cdots$, if $\gcd(2,m-j)=1$; or two smooths branches g_1,g_2 with parametrizations $x_1=t,\ y_1=\sqrt{2}t^{\frac{m-j}{2}}+\cdots$ and $x_2=t,\ y_2=-\sqrt{2}t^{\frac{m-j}{2}}+\cdots$, if $\gcd(2,m-j)=2$.

b) Associated to L_2, there is one smooth branch p_2 with parametrization $x=t,\ y=-\frac{am}{8b}t^{j-1}+\cdots$.

Finally, we have that $\mathrm{I}(p_1,p_2)=m-j$ and $\mathrm{I}(g_i,p_2)=\mathrm{I}(g_1,g_2)=\frac{m-j}{2}$.

iii. Case $\frac{2}{m-j}=\frac{1}{j-1}$.

Since $j>2$, because otherwise $m=4$, which is not allowed, the Newton polygon of the polar has only one side L with tree points and the polynomial associated to L is $p_L(z)=4bz^3-8bz-am$. Therefore, for a and b generic, the polynomial $p_L(z)$ has three distinct roots $\{z_1,z_2,z_3\}$ and as, in this case, $\gcd(3,m-1)=3$, then associated to L we have three smooth branches with parametrizations $(t,z_it^{j-1}+\cdots)$, $i=1,2,3$, and mutual intersection numbers $j-1$.

(II) Now we consider the case where some of the a_i's is non-zero. Set $k=\min\{i;\ a_i\neq 0\}$.

After a computation we get

$$f=y^4-(4x^{m-j}+2a_k^2x^{m-2(j-[\frac{m}{4}]-k)}u_1)y^2-4a_kx^{m-(j-[\frac{m}{4}]-k)}u_2y-x^mu_3,$$

where $u_i\in\mathbb{C}\{x\}$ with $u_i(0)=1$ for $i=1,2,3$. Hence, to determine the Newton polygon of the polar af_x+bf_y, it is sufficient to consider the polynomial

$$4by^3-4a(m-j)x^{m-j-1}y^2-8bx^{m-j}y-4ba_kx^{m-j+[\frac{m}{4}]+k}.$$

We now split the analysis of this case into several sub-cases.

i. Case $\frac{2}{m-j} < \frac{1}{[\frac{m}{4}]+k}$.

The Newton polygon of the polar has just one side L, containing only the points $(0,3)$ and $(m-j+[\frac{m}{4}]+k,0)$.

Since the polynomial $p_L(z) = 4bz^3 - 4ba_k$ has three distinct roots $\{z_1, z_2, z_3\}$, it follows that the polar has:

a) Only one branch, if $\gcd(3, m-j+[\frac{m}{4}]+k) = 1$, with semigroup $\langle 3, m-j+[\frac{m}{4}]+k\rangle$.

b) Three smooth branches, if $\gcd(3, m-j+[\frac{m}{4}]+k) = 3$, with parameterizations

$$x_i = t, \quad y_i = z_i t^{\frac{m-j+[\frac{m}{4}]+k}{3}} + \cdots, \qquad i \in \{1,2,3\},$$

and mutual intersection numbers $\frac{m-j+[\frac{m}{4}]+k}{3}$.

ii. Case $\frac{2}{m-j} > \frac{1}{[\frac{m}{4}]+k}$.

In this case, the Newton polygon of the polar has two sides $L_1 = [(0,3);(m-j,1)]$ and $L_2 = [(m-j,1);(m-j+[\frac{m}{4}]+k,0)]$, with on each side only the extreme points correspond to monomials of the polar.

Considering the polynomials associated to these sides, $p_{L_1}(z) = 4bz^2 - 8b$ and $p_{L_2}(z) = -8bz - 4ba_k$; and defining $d = \gcd(2, m-j)$, we have that

a) Associated to the side L_1, we have a branch p_1 with semigroup $\langle 2, m-j\rangle$ and parametrization $x = t^2, \ \ y = \sqrt{2}t^{m-j} + \cdots$, if $d = 1$; and two smooth branches g_1, g_2, with parametrizations $x_i = t, \ \ y_i = (-1)^{i-1}\sqrt{2}t^{\frac{m-j}{2}} + \cdots$, $i = 1,2$, if $d = 2$.

b) Associated to the side L_2, we have a smooth branch p_2, with parametrization $x = t, \ \ y = -\frac{a_k}{2}t^{[\frac{m}{4}]+k} + \cdots$.

Finally, one has $\mathrm{I}(p_1,p_2) = m-j$ and $\mathrm{I}(g_i,p_2) = \mathrm{I}(g_1,g_2) = \frac{m-j}{2}$.

iii. Case $\frac{2}{m-j} = \frac{1}{[\frac{m}{4}]+k}$.

In this case, the Newton polygon of the polar has a unique side L containing the three points $(0,3),(m-j,1)$ and $(m-j+[\frac{m}{4}]+k,0)$. whose associated polynomial is

$$p_L(z) = 4bz^3 - 8bz - 4a_k.$$

When $a_k \neq \frac{4\sqrt{6}}{9}(-1)^\alpha; \ \alpha = 0,1$, because of the condition $\frac{2}{m-j} = \frac{1}{[\frac{m}{4}]+k}$, it is easy to verify that the polar is Newton non-degenerate. In this case, the polynomial $p_L(z)$ has three distinct roots $\{z_1,z_2,z_3\}$, then the polar has three smooth branches with parametrizations $x = t$ and $y_i = z_i t^{[\frac{m}{4}]+k} + \cdots$, $i = 1,2,3$, with mutual intersection numbers equal to $[\frac{m}{4}]+k = \frac{m-j}{2}$.

Now we suppose that $a_k = \frac{4\sqrt{6}}{9}(-1)^\alpha; \ \alpha = 0,1$.

In this case, the roots of $p_L(z)$ are $\frac{\sqrt{6}}{3}(-1)^{\alpha+1}$, $\frac{\sqrt{6}}{3}(-1)^{\alpha+1}$ and $2\frac{\sqrt{6}}{3}(-1)^{\alpha}$. The polar will have a smooth branch f_1 corresponding to the simple root of $p_L(z)$ and branches g_i corresponding to the double root.

We may suppose that the roots of $p_L(z)$ are $\frac{\sqrt{6}}{3}$, $\frac{\sqrt{6}}{3}$ and $-2\frac{\sqrt{6}}{3}$, since the other case is analogous.

a) If for all $l > 0$ one has $a_{k+l} = 0$, then a simple analysis shows that the polar has a smooth branch and a branch with semigroup $\langle 2, 2m-3j\rangle$ with intersection number $m-j$.

b) Suppose that there exists $l > 0$ such that $a_{k+l} \neq 0$. We denote the least such l by s. In this case, we will need in our analysis to consider more terms of f, which now reads as

$$
\begin{aligned}
f \;=\;& y^4 + (-4x^{m-j} - 2a_k^2 x^{m-2(j-[\frac{m}{4}]-k)} - 4a_k a_{k+s} x^{m-2(j-[\frac{m}{4}]-k)+s} \\
& -2a_{k+s}^2 x^{m-2(j-[\frac{m}{4}]-k-s)} + \cdots)y^2 - (4a_k x^{m-j+[\frac{m}{4}]+k} \\
& +4a_{k+s} x^{m-j+[\frac{m}{4}]+k+s} + 4a_k x^{2m-2j-(j-[\frac{m}{4}]-k)} \\
& +4a_{k+s} x^{2m-2j-(j-[\frac{m}{4}]-k-s)} + \cdots)y - ux^m,
\end{aligned}
$$

where $u \in \mathbb{C}\{x\}$ with $u(0) = 1$.

Now, in order to apply the Newton-Puiseux algorithm to the general polar of f at the double root of $p_L(z)$, we have to split our analysis in several subcases.

b.1) $m - 2j > s$.

b.1.1) s odd. Associated to the double root there is a branch g_1 given by

$$x = t^2,\; y = -\frac{\sqrt{6}}{3} t^{m-j} + \frac{\sqrt{a_{k+s}}}{\sqrt[4]{6}} t^{m-j+s} + \cdots.$$

In this case, the polar has a smooth branch f_1 and a branch g_1 with semigroup $\langle 2, m-j+s\rangle$ such that $\mathrm{I}(f_1, g_1) = m - j$.

b.1.2) s even. The polar splits into three smooth factors f_1, g_1 and g_2, such that $\mathrm{I}(f_1, g_i) = [\frac{m}{4}] + k$ and $\mathrm{I}(g_1, g_2) = [\frac{m}{4}] + k + \frac{s}{2}$.

b.2) $m - 2j < s$. In this case, the polar has the smooth branch f_1 and a branch branch g_1 associated to the double root with semigroup $\langle 2, 2m-3j\rangle$, such that $\mathrm{I}(f_1, g_1) = m - j$.

b.3) $m - 2j = s$.

b.3.1) If $a_{k+s} \neq \frac{4\sqrt{6}}{81}(-1)^{\alpha+1}$, we have, associated to the double root, a branch g_1 with semigroup $\langle 2, m-j+s\rangle$. So, the polar has the smooth branch f_1 and the above branch g_1 such that $\mathrm{I}(f_1, g_1) = m - j$.

b.3.2) If $a_{k+s} = \frac{4\sqrt{6}}{81}(-1)^{\alpha+1}$, we have, associated to the double root, two smooth branches g_1 and g_2 such that $\mathrm{I}(f_1, g_i) = \frac{m-j}{2}$, $i = 1, 2$, and $\mathrm{I}(g_1, g_2) = \frac{m-j}{2} + s$.

The following table summarizes the above analysis for the second normal form.

$y = t^m + t^{3m-4j} + a_1 t^{2m-4(j-[\frac{m}{4}]-1)} + \cdots + a_{j-[\frac{m}{4}]-2} t^{2m-8};\ 2 \le j \le [\frac{m}{2}]$	
$a_1 = a_2 = \cdots = a_{j-[\frac{m}{4}]-2} = 0$	
$\frac{2}{m-j} < \frac{1}{j-1}$	The polar has one branch with semigroup $\langle 3, m-1 \rangle$, if $\gcd(3, m-1) = 1$; otherwise it has three smooth branches p_1, p_2, p_3, with $\mathrm{I}(p_i, p_r) = \frac{m-1}{3}$.
$\frac{2}{m-j} > \frac{1}{j-1}$	The polar has one branch p_1 with semigroup $\langle 2, m-j \rangle$ and one smooth branch p_2, with $\mathrm{I}(p_1, p_2) = m-j$, if $\gcd(2, m-j) = 1$; otherwise it has three smooth branches p_1, p_2, p_3, with $\mathrm{I}(p_i, p_r) = \frac{m-j}{2}$.
$\frac{2}{m-j} = \frac{1}{j-1}$	The polar has three smooth branches p_1, p_2, p_3, with $\mathrm{I}(p_i, p_r) = j-1$.
$\exists\, i;\ a_i \neq 0, \qquad k = \min\{i;\ a_i \neq 0\}$	
$\frac{2}{m-j} < \frac{1}{[\frac{m}{4}]+k}$	The polar has one branch with semigroup $\langle 3, m-j+[\frac{m}{4}]+k \rangle$, if $\gcd(3, m-j+[\frac{m}{4}]+k) = 1$; otherwise it has three smooth branches p_1, p_2, p_3 with $\mathrm{I}(p_i, p_r) = \frac{m-j+[\frac{m}{4}]+k}{3}$.
$\frac{2}{m-j} > \frac{1}{[\frac{m}{4}]+k}$	The polar has a branch p_1, with semigroup $\langle 2, m-j \rangle$ and a smooth branch p_2, with $\mathrm{I}(p_1, p_2) = m-j$, if $\gcd(2, m-j) = 1$; otherwise it has three smooth branches p_1, p_2, p_3, with $\mathrm{I}(p_i, p_r) = \frac{m-j}{2}$.
$\frac{2}{m-j} = \frac{1}{[\frac{m}{4}]+k}$	For $a_k \neq \frac{4\sqrt{6}}{9}(-1)^\alpha$; $\alpha = 0, 1$, the polar has three smooth branches p_1, p_2, p_3, with $\mathrm{I}(p_i, p_r) = \frac{m-j}{2}$. For $a_k = \frac{4\sqrt{6}}{9}(-1)^\alpha$; $\alpha = 0, 1$: a) If $a_{k+l} = 0$, $\forall l > 0$, then the polar has a smooth branch f_1 and a branch g_1 with semigroup $\langle 2, 2m-3j \rangle$ with $\mathrm{I}(f_1, g_1) = m-j$. b) There exists $s > 0$ such that $a_{k+s} \neq 0$ (let s be minimum). b.1) $m - 2j > s$. b.1.1) s odd. The polar has a smooth branch f_1 and a branch g_1 with semigroup $\langle 2, m-j+s \rangle$ with $\mathrm{I}(f_1, g_1) = m-j$. b.1.2) s even. The polar has three smooth branches f_1, g_1, g_2 with $\mathrm{I}(f_1, g_i) = \frac{m-j}{2}$ and $\mathrm{I}(g_1, g_2) = \frac{m-j+s}{2}$. b.2) $m - 2j < s$. The polar has a smooth branch f_1 and a branch g_1 with semigroup $\langle 2, 2m-3j \rangle$ with $\mathrm{I}(f_1, g_1) = m-j$. b.3) $m - 2j = s$. b.3.1) $a_{k+s} \neq \frac{4\sqrt{6}}{81}(-1)^{\alpha+1}$. The polar has a smooth branch f_1 and a branch g_1 with semigroup $\langle 2, 2m-3j \rangle$ with $\mathrm{I}(f_1, g_1) = m-j$. b.3.2) $a_{k+s} = \frac{4\sqrt{6}}{81}(-1)^{\alpha+1}$. The polar has three smooth branches f_1, g_1, g_2 such that $\mathrm{I}(f_1, g_i) = \frac{m-j}{2}$ and $\mathrm{I}(g_1, g_2) = \frac{m-j}{2} + s$.

Table 3.2: The polars for curves in the second Normal Form

Third to Fifth Normal Forms

These are simple to analyze and all give the same result, summarized in the following table:

$\gcd(3, m-j) = 1$	One branch with semigroup $\langle 3, m-j \rangle$.
$\gcd(3, m-j) = 3$	Three smooth branches with mutual intersection numbers $\frac{m-j}{3}$.

Table 3.3: The polars for curves in the third to fifth Normal Forms

3.3 Multiplicity four and genus two

The classification of multiplicity 4 and genus 2 branches is given in the table below, extracted from [HH2].

Normal form	$\Lambda \setminus \langle 4, v_1, v_2\rangle$
$y(t) = t^{v_1} + t^{v_2 - v_1} + a_1 t^{v_2 - 4[\frac{v_1}{4}]}$ $+ a_2 t^{v_2 - 4([\frac{v_1}{4}]-1)} + \cdots + a_{[\frac{v_1}{4}]-1} t^{v_2 - 8}$,	$v_2 + v_1 - 4s$; $1 \le s \le \frac{v_1}{2} + 1$

Table 3.4: Multiplicity four and genus two

Since $\gcd(4, v_1, v_2) = 1$ and $v_2 > 2v_1$, we may write $v_1 = 2k_1$ and $v_2 = 4k_1 + d$, where k_1 and d are odd numbers. It is easy to verify that $2v_2 - v_1$ is multiple of 4, let us write $k_2 = \frac{2v_2 - v_1}{4}$.

Writing an implicit equation for the curve, we have

$$f = y^4 + (-2x^{k_1} + \cdots)y^2 + (-4x^{k_2} + \cdots)y + x^{v_1}u,$$

where $u \in \mathbb{C}\{x\}$ and $u(0) = 1$.

The Newton polygon of the polar $af_x + bf_y$ is determined by the polynomial

$$4by^3 - 2k_1 a x^{k_1 - 1} y^2 - 4bx^{k_1} y - 4bx^{k_2} + v_1 a x^{v_1 - 1}.$$

This polygon has always two sides L_1 and L_2 whose positions depend upon the relationship among k_2 and $v_1 - 1$. In all cases, the polar will have a component g_1 associated to side L_1 given by the following parametrization

$$x_1 = t^2, \;\; y_1 = t^{k_1} + \cdots .$$

Associated to the side L_2, the polar has a branch g_2 parametrized by

$$x_2 = t, \quad y_2 = \begin{cases} \frac{ak_1}{2b} t^{v_1 - k_1 - 1} + \cdots , & \text{if } v_1 - 1 < k_2, \\ -\frac{2b - k_1 a}{2b} t^{v_1 - k_1 - 1} + \cdots , & \text{if } v_1 - 1 = k_2, \\ -t^{k_2 - k_1} + \cdots , & \text{if } v_1 - 1 > k_2, \end{cases}$$

with $\mathrm{I}(g_1, g_2) = k_1$.

In the sequel we give an example of a curve (f) for which the analytic type of its polar curve $(af_x + bf_y)$ depends essentially on the direction $(a : b)$.

Example 2. Consider the curve (f) given parametrically by $(t^5, t^{12} + t^{21})$, that belongs to the eighth family in Example 1. We know that, in this case, $af_x + bf_y = 5by^4 - 10bux^9 y - 12avx^{11}$, where u and v are units in $\mathbb{C}\{x\}$. This polar is irreducible and is analytically equivalent to a branch with parametrization

$$\left(t^4, t^{11} + t^{14} - \frac{1}{2}t^{17} + \frac{15\sqrt[3]{2}}{2}\big(\frac{12a}{5b}\big)^3 t^{21}\right).$$

This is a branch of multiplicity four belonging to the fourth Normal Form in Table 3.1. So from the Normal Forms theorem, two such branches corresponding to directions $(a : b)$ and $(a' : b')$ with $bb' \neq 0$ are analytically equivalent if and only if one has $\frac{a^3}{b^3} = \frac{a'^3}{b'^3}$.

As a final remark, we refer to [MP] for a rough description of the polars of the members of the equisingularity class determined by the semigroup $\langle 5, 11 \rangle$, which could be completely described by the methods we exhibited in the present paper.

References

[BLP] García-Barroso, E. R.; Lenarcik, A.; Ploski, A. - *Characterization of non-degenerate plane curve singularities.* Proc. of the Effective Methods in Algebraic and Analytic Geometry. Effect 2006. Univ. Iagell. Acta Math. Fasc. XLV (2007), 27-36.

[C2] Casas-Alvero, E. - *Infinitely near imposed singularities and singularities of polar curves.* Math. Ann. 287 (1990), 429-454.

[C3] Casas-Alvero, E. - *Singularities of Plane Curves.* London Math. Soc. Lecture Notes Series 276 (2000).

[HH1] Hefez, A.; Hernandes, M.E. - *Standard bases for local rings of branches and their module of differentials.* J. Symb. Comput. 42 (2007), 178-191.

[HH2] Hefez, A.; Hernandes, M.E. - *Analytic classification of plane branches up to multiplicity 4.* J. Symb. Comput. 44 (2009), 626-634.

[HH3] Hefez, A.; Hernandes, M.E. - *The analytic classification of plane branches.* Bull. London Math. Soc. 43 (2011), 289-298.

[HH4] Hefez, A.; Hernandes, M.E. - *Algorithms for the implementation of the analytic classification of plane branches.* J. Symb. Comput. 50 (2013), 308-313.

[HI] Hernández Iglesias, M.F. - *Polar de um germe de curva irredutível plana.* Tese de Doutorado, UFF (2012).

[K] Kouchnirenko, A.G. - *Polyèdres de Newton et nombres de Milnor.* Inv. Math. 32 (1976), 1-31.

[LR] Lê D. T.; Ramanujan, C. P. - *The invariance of Milnor's numbers implies the invariance of the topological type.* American Journal of Mathematics, Vol 98 (1976), 67-78.

[MP] MARTIN, B.; PFISTER, G. - *The moduli of irreducible curve singularities with the semigroup* $\langle 5, 11 \rangle$. Rev. Roumaine XXXIII (4) (1988), 359-368.

[O] OKA, M. - *Non-degenerate complete intersection singularity*. Actualités Mathématiques, Hermann (1997).

[P] PHAM F. - *Deformations equisingulières des idéaux jacobiens des courbes planes*. In Proc of Liverpool Symposium on Singularities II, Volume 209 of Lect Notes in Math, Springer Verlag, Berlin, London, new york (1971), 218-233.

[Z] ZARISKI, O. - *The moduli problem for plane branches*. University lecture series AMS, Volume 39 (2006).